# ON THE ROUTH THEOREM AND THE CHETAEV METHOD FOR CONSTRUCTING THE LIAPUNOV FUNCTION FROM THE INTEGRALS OF THE EQUATIONS OF MOTION

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The first studies on the stability of steady motions were carried out by Routh [1, 2]. Routh's method is based on his fundamental theorem [2], which was considerably supplemented by Liapunov [3]. The stability of steady motions can also be studied on the basis of the Liapunov stability theorem [4]; the latter involves constructing the Liapunov function from the integrals of the equations of motion by the Chetaev method [4–6]. These two approaches are basic to the investigation of the stability of steady motions.

We intend to show that under sufficiently broad assumptions the same stability conditions are obtainable by using the Routh theorem on the one hand and the Chetaev method for constructing the Liapunov function (in the case of a complete bundle of integrals) on the other. A very similar result is arrived at in [7]. Bifurcation of steady motions is considered. Two theorems on the nominal sign constancy of the quadratic forms are noted. The suitability of the Routh theorem for finding the steady motions and the conditions of their stability for a rigid body with a cavity completely filled with an ideal or viscous fluid is demonstrated (stability is established with respect to the parameters characterizing the state of motion of the body and with respect to certain other parameters characterizing the motion of the fluid in integral fashion). This formulation of the problem of stability of motion of a fluid-filled body was proposed by Rumiantsev [6], who solved it by constructing the Liapunov function by the Chetaev method in the form of a bundle of integrals of the equations of motion. Unlike Arnol'd (e.g. [8]), who investigated the stability of steady motions of a fluid, we shall analyze the stability of the steady motions of the body-fluid system with respect to a finite number of parameters.

**1.** Liapunov formulated the Routh theorem with his own addenda in the following way [3].

When a certain number of time-independent integrals has been obtained for the differential equations of motion of some system, and when among these integrals there exists one which can have a minimum or maximum for given values of the other integrals, assuming this minimum or maximum value for certain values of the variables occurring in it, then these values in general correspond to one of the real motions of the system; moreover, this motion is stable with respect to the variables in question at least for perturbations which do not alter the values of the other integrals. If the integral in question also has a minimum or maximum for all values of the other integrals sufficiently close to the given values and if the values of the variables which minimize or maximize this integral are continuous functions of the values of these integrals, then the motion in question is stable for all perturbations.

Liapunov did not prove this theorem. However, his later theorem on stability [4] has been used to show [7, 9] that in the case of continuous integrals verification of the Liapunov requirement of unconditional stability is not necessary.

Theorem 1. Let the equations of motion of some system have the time-independent integrals

$$U_0(x_1, x_2, \ldots, x_n) = c_0, \quad U_1(x_1, x_2, \ldots, x_n) = c_1, \ldots, \quad U_k(x_1, x_2, \ldots, x_n) = c_k \quad (1.1)$$

continuous with respect to the quantities $x_1, x_2, \ldots, x_n$, and let $U_0$ have the isolated minimum (maximum)

$$U_0(x_1^\circ, x_2^\circ, \ldots, x_n^\circ) = c_0^\circ$$

for fixed values of the remaining integrals,

$$U_1(x_1, x_2, \ldots, x_n) = c_1^\circ, \ldots, U_k(x_1, x_2, \ldots, x_n) = c_k^\circ \quad (1.2)$$

assuming the minimum (maximum) for the values $x_1^\circ, x_2^\circ, \ldots, x_n^\circ$ of its variables. These values then correspond to one of the real motions of the system , and the motion is stable with respect to $x_1, x_2, \ldots, x_n$.

Proof. Let us consider the system motion in which $x_1 = x_1^\circ$, $x_2 = x_2^\circ, \ldots, x_n = x_n^\circ$ at the instant $t^\circ$. In addition, let us assume that $x_1 = x_1^*, x_2 = x_2^*, \ldots, x_n = x_n^*$ at some instant $t^*$ and that at least one of the values $x_1^*, x_2^*, \ldots, x_n^*$ does not coincide with the corresponding value $x_1^\circ, x_2^\circ, \ldots, x_n^\circ$. Then, by definition of the integral, we have

$$U_0(x_1^*, x_2^*, \ldots, x_n^*) = U_0(x_1^\circ, x_2^\circ, \ldots, x_n^\circ) = c_0^\circ$$

Since $c_0^\circ$ is the minimum (maximum) of $U_0$ under conditions (1.2), it must be the case that

$$U_0(x_1^*, x_2^*, \ldots, x_n^*) > U_0(x_1^\circ, x_2^\circ, \ldots, x_n^\circ) = c_0^\circ \quad (U_0(x_1^*, x_2^*, \ldots, x_n^*) < c_0^\circ)$$

which contradicts the preceding equation. Hence, the quantities $x_1, x_2, \ldots, x_n$ in the motion under consideration retain their constant values $x_1^\circ, x_2^\circ, \ldots, x_n^\circ$.

Further, let us consider the function [10, 7, 9]

$$\Phi(y_1, y_2, \ldots, y_n) = \sum_{i=0}^{k} V_i^2, \quad V_i = U_i - c_i^\circ, \quad i = 0, 1, \ldots, k$$

By the conditions of theorem, this function is positive-definite with respect to the perturbations $y_i = x_i - x_i^\circ$ $(i = 1, 2, \ldots, n)$ and is an integral of the system and thereby satisfies the requirements of the Liapunov stability theorem [4]. The theorem has been proved.

Stability can also be proved [10, 11] by reasoning similar to that of the Lejeune-Dirichlet proof of the Lagrange theorem.

Note 1. The form of the equations of motion of the system under investigation is not used anywhere in the above proof. Hence, the Routh theorem can be used to investigate the steady motions not only of systems with a finite number of degrees of freedom, but also of systems with infinitely many degrees of freedom.

Note 2. The above proof remains valid when $U_0$ is not an integral, but rather some function (or functional) which does not increase (does not decrease) along the system motions [12].

Note 3. Theorem 1 is also valid when the variables $x_1, x_2, \ldots, x_n$ are restricted by relations of the form

$$F(x_1, x_2, \ldots, x_n) = 0$$

In this case the restrictions must be taken into account along with conditions (1.2) in determining the minimum (maximum) of $U_0$ These relations can represent either holonomic constraints [12, 13], or particular integrals of the system (in the latter case only nominal stability can be established).

Note 4. In the case of conservative systems with cyclical coordinates the requirements of Theorem 1 reduce to the requirement that the altered potential energy of the system be minimal [1].

**2.** Let us consider the Chetaev method and its relationship to the Routh theorem. Let the parameter values
$$x_1 = x_1^\circ, \ x_2 = x_2^\circ, \ \ldots, \ x_n = x_n^\circ$$
correspond to some steady motion of the system. In the Chetaev method the Liapunov function for this motion must be determined in the form of a bundle of integrals,

$$V = V_0 + \sum_{i=1}^{k} \lambda_i V_i + \sum_{i=1}^{k} \mu_i V_i^2 \tag{2.1}$$
$$V_i = U_i(x_1, x_2, \ldots, x_n) - U_i(x_1^\circ, x_2^\circ, \ldots, x_n^\circ)$$

If the constants $\lambda_1, \lambda_2, \ldots, \lambda_k, \mu_1, \mu_2, \ldots, \mu_k$ can be chosen in such a way that $V$ is a function of constant sign with respect to $y_i = x_i - x_i^\circ (i = 1, 2, \ldots, n)$ , then the steady motion is stable with respect to $x_1, x_2, \ldots, x_n$ by virtue of the Liapunov stability theorem. If (2.1) does not contain squares of some of the integrals $V_1, V_2, \ldots, V_k$, then the bundle of integrals will be called "incomplete".

Now let us assume that integrals (1.1) are continuously differentiable twice and consider the basic case where the conditions of positive-definiteness of the function $V$ and of the nominal minimum of $U_0$ (the case of negative-definiteness of $V$ and of the nominal maximum of $U_0$ are reducible to the same case by changing the sign of the integral $U_0$) can be formulated as the Sylvester conditions of sign constancy or nominal sign constancy [14] of certain quadratic forms.

Theorem 2. Under the above conditions the Routh theorem and the Chetaev method of constructing the Liapunov function in the form of a complete bundle of integrals yield the same stability conditions provided the same integrals are used.

Proof. Let us compare the stability conditions obtainable from Theorem 1 and by means of the Chetaev method.

The values of the variables $x_1, x_2, \ldots, x_n$ for which $U_0$ has stationary values under conditions (1.2) can be determined by the method of indefinite Lagrange multipliers from the equations

$$\frac{\partial W}{\partial \lambda_1} = U_1 - c_1^\circ = 0, \ldots, \frac{\partial W}{\partial \lambda_k} = U_k - c_k^\circ = 0, \ \frac{\partial W}{\partial x_1} = 0, \ldots, \frac{\partial W}{\partial x_n} = 0 \tag{2.2}$$
$$W = U_0 + \lambda_1 (U_1 - c_1^\circ) + \ldots + \lambda_k (U_k - c_k^\circ)$$

Let
$$\lambda_1 = \lambda_1^\circ, \ldots, \lambda_k = \lambda_k^\circ, \quad x_1 = x_1^\circ, \ldots, x_n = x_n^\circ \tag{2.3}$$
be a solution of system (2.2) and let us introduce the notation
$$A = \| a_{ij} \|_{i,\, j=1}^{i,\, j=n} = \left\| \frac{\partial^2 W}{\partial x_i \partial x_j} \right\|, \quad B = \| b_{ij} \|_{i,\, j=1}^{i=k,\, j=n} = \left\| \frac{\partial U_i}{\partial x_j} \right\|$$
where the derivatives must be computed for values (2.3). We assume that
$$\det \| b_{ij} \|_{i,\, j=1}^{i,\, j=k} \neq 0 \tag{2.4}$$

The values $x_1^\circ, x_2^\circ, \ldots, x_n^\circ$ correspond to the nominal minimum of the integral $U_0$ if the quadratic form
$$(Ay, y) = \sum_{i,\, j=1}^{n}{}' a_{ij} y_i y_j \tag{2.5}$$

is positive-definite under the conditions

$$(By)_i = b_{i1}y_1 + b_{i2}y_2 + \ldots + b_{in}y_n = 0, \quad i = 1, 2, \ldots, k \tag{2.6}$$

Let us consider the determinant

$$\Delta = (-1)^k \begin{vmatrix} \theta & B \\ B^\tau & A \end{vmatrix} \tag{2.7}$$

where $\theta$ is a $(k \times k)$ null matrix and where the symbol $\tau$ denotes transposition. This determinant is equal to within $(-1)^k$ to the Jacobian of system (2.2) and to the Hessian $W$ in $\lambda_1, \lambda_2, \ldots, \lambda_k, x_1, x_2, \ldots, x_n$ for values (2.3). Quadratic form (2.6) is positive-definite under conditions (2.6) if and only if [14] the principal diagonal minors of the determinant $\Delta$ starting with the minor of order $2k + 1$ are positive, i.e. if

$$\Delta_{k+1} > 0, \ldots, \Delta_n > 0 \tag{2.8}$$

where $\Delta_i$ is the principal diagonal minor of order $k + i$. Thus, fulfillment of conditions (2.8) ensures fulfillment of the requirements of Theorem 1.

The constants $\lambda_1, \lambda_2, \ldots, \lambda_k$ in the Chetaev method must be chosen in such a way as to eliminate the linear part of bundle (2.1); they obviously coincide with values (2.3). Moreover,

$$V = (Cy, y) + o(|y|^2), \qquad \lim_{s \to 0} o(s)/s = 0$$

$$(Cy, y) = (Ay, y) + \sum_{\nu=1}^{k} \mu_\nu (By)_\nu^2, \quad |y|^2 = y_1^2 + \ldots + y_n^2 \tag{2.9}$$

It is easy to show that the principal diagonal minors $D_1, D_2, \ldots, D_n$ of the discriminant of the quadratic form $(Cy, y)$ are equal to the respective principal diagonal minors, beginning with the minor of order $k + 1$, of the determinant $D$,

$$D = (-1)^k \mu_1 \ldots \mu_k \begin{vmatrix} Q & B \\ B^\tau & A \end{vmatrix}, \quad Q = -\operatorname{diag}(\mu_1^{-1}, \ldots, \mu_k^{-1})$$

The Sylvester conditions of positive-definiteness of quadratic form (2.9) can be expressed in powers $\mu_1, \mu_2, \ldots, \mu_k$ as follows:

$$D_1 = \mu_1 b_{11}^2 + \mu_2 b_{12}^2 + \ldots + \mu_k b_{1k}^2 + a_{11} > 0 \tag{2.10}$$

$$D_\nu = \sum_{\alpha_1 < \ldots < \alpha_\nu = 1}^{k} \mu_{\alpha_1} \ldots \mu_{\alpha_\nu} \begin{vmatrix} b_{\alpha_1 1} & \ldots & b_{\alpha_1 \nu} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ b_{\alpha_\nu 1} & \ldots & b_{\alpha_\nu \nu} \end{vmatrix}^2 + \sum_{\alpha_1 < \ldots < \alpha_{\nu-1} = 1}^{k} \mu_{\alpha_1} \ldots \mu_{\alpha_{\nu-1}} \Delta_\nu^{\alpha_1 \ldots \alpha_{\nu-1}} +$$

$$+ \sum_{\alpha_1 < \ldots < \alpha_{\nu-2} = 1}^{k} \mu_{\alpha_1} \ldots \mu_{\alpha_{\nu-2}} \Delta_\nu^{\alpha_1 \ldots \alpha_{\nu-2}} + \ldots + \Delta_\nu^\circ > 0 \quad (\nu = 2, 3, \ldots, k)$$

$$D_\varkappa = \mu_1 \mu_2 \ldots \mu_k \Delta_\varkappa + \sum_{\alpha_1 < \ldots < \alpha_{k-1} = 1}^{k} \mu_{\alpha_1} \ldots \mu_{\alpha_{k-1}} \Delta_\varkappa^{\alpha_1 \ldots \alpha_{k-1}} +$$

$$+ \sum_{\alpha_1 < \ldots < \alpha_{k-2} = 1}^{k} \mu_{\alpha_1} \ldots \mu_{\alpha_{k-2}} \Delta_\varkappa^{\alpha_1 \ldots \alpha_{k-2}} + \ldots + \Delta_\varkappa^\circ > 0 \quad (\varkappa = k + 1, \ldots, n)$$

Here $\Delta_\sigma^{\alpha_1 \ldots \alpha_\rho}$ are the principal diagonal minors of the order $\sigma + \rho$ of the determinant $\Delta^{\alpha_1 \ldots \alpha_\rho}$ similar to $\Delta$, but satisfying only some of conditions (2.6) (those with the numbers $\alpha_1, \alpha_2, \ldots, \alpha_\rho$), and

$$\Delta_\sigma^\circ = \det \| a_{ij} \|_{i,\, j=1}^{i,\, j=\sigma}$$

On fulfillment of conditions (2.8) and sufficiently large $\mu_1, \mu_2, \ldots, \mu_k$ conditions (2.10) are fulfilled by virtue of (2.8) and (2.4). Quadratic form (2.9) and the function $V$ are then positive-definite. Conversely, if conditions (2.10) are fulfilled for some $\mu_1, .., \mu_k$, then quadratic form (2.9) is positive-definite, and especially positive-definite under those conditions (2, 6) for which (2.9) coincides with (2.5). Conditions (2.8) are then fulfilled by virtue of their necessity. Theorem 2 has been proved.

The following theorem on the sign constancy of the quadratic forms reflects the relationship between the Chetaev method and the Routh theorem. It is also related to the "method of penalty functions".

Theorem 3. Quadratic form (2.5) is positive-definite under conditions (2.6) if and only if quadratic form (2.9) is positive-definite for sufficiently large $\mu_1, \mu_2, \ldots, \mu_k$ (*).

Theorem 3 implies that if the perturbations $y_1, y_2, \ldots, y_n$ must satisfy relations of the form

$$F(y_1, y_2, \ldots, y_n) = 0 \qquad (F(0, 0, \ldots, 0) = 0) \tag{2.11}$$

then the left sides of these relations can be included in bundle (2.1) along with the integrals $V_1, V_2, \ldots, V_k$. The resulting stability conditions are equivalent to those obtainable either by eliminating the dependent variables or by exploiting the positive-definiteness of bundle (2.1); moreover, if relations of the form (2.11) express the holonomic or nonholonomic constraints imposed on the system (e. g. if they are particular integrals of the system), then the resulting stability is nominal.

Note 5. In computing determinants (2.8) it is sometimes expedient to use the Laplace theorem on determinants, expanding them in the first $k$ rows and columns.

**3.** Conditions (2.8) of nominal sign constancy of quadratic form (2.5) remain valid if we replace assumption (2.4) by the weaker requirement [15]

$$\operatorname{rang} \| b_{ij} \|_{i=1,\ j=1}^{i=k,\ j=k+1} = k$$

However, this assumption can also occasion certain difficulties in studying stability domains in the parameter space.

We note that conditions (2.8) are symmetric in $y_1, y_2, \ldots, y_n$ and assume only that Eqs. (2.6) are independent.

Theorem 4. Quadratic form (2.5) is positive-definite under independent conditions (2.6) if and only if

$$S_{k+1} > 0, \ldots, S_n = \Delta > 0 \tag{3.1}$$

where the $S_i$ represent the sums of all the possible minors of order $k+i$ of determinant (2.7) which border its principal diagonal minor of order $k$ (consisting of zero elements).

Proof. Let us introduce the $n$-dimensional Euclidean space $R$ with the orthogonal axes $x_1, x_2, \ldots, x_n$ and denote its $(n-k)$-dimensional subspace defined by Eqs. (2.6) by $Q$. Next, let us reduce quadratic form (2.5) on subspace $Q$ to canonical form. To this end we determine its fixed values on a unit sphere

$$(x, x) = x_1^2 + x_2^2 + \ldots + x_n^2 = 1 \tag{3.2}$$

in the subspace $Q$. The equations for determining the fixed points of quadratic form (2.5) under conditions (2.6) and (3.2) can be expressed in terms of the Lagrange multipliers $\sigma, \lambda_1, \lambda_2, .., \lambda_k$ in the form

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*) After submitting the present paper for publication we learned that a similar statement has already been formulated by Prof. P. A. Kuz'min.

$$\partial F/\partial x_i = 0 \quad (i = 1, 2, \ldots, n), \quad 2F = (Ax, x) - \sigma(x, x) + 2\sum_{j=1}^{k} \lambda_j (Bx)_j \tag{3.3}$$

The determinant of subsystem (2.6), (3.3) linear in $\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_n$ must vanish for the solutions of system (2.6), (3.3), (3.2),

$$D^*(\sigma) = \begin{vmatrix} \theta & B \\ B^\tau & A - \sigma E_n \end{vmatrix} = \sum_{\nu=0}^{n-k} (-1)^{n+\nu} a_\nu \sigma^{n-k-\nu} = 0 \tag{3.4}$$

Here

$$a_0 = S_k, \; a_1 = S_{k+1}, \ldots, a_{n-k} = S_n = \Delta \tag{3.5}$$

As in the case of the Sylvester theorem we can show that all the roots $\sigma_1, \sigma_2, \ldots, \sigma_{n-k}$ of Eq. (3.4), which is a natural generalization of the secular equation, are real and equal to the required fixed values of form (2.5) on sphere (3.2).

In the axes $z_1, z_2, \ldots, z_{n-k}$ whose orthogonal normed basis is defined by the solutions of system (3.2), (2.6), (3.3) for $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_{n-k}$ quadratic form (2.5) on the subspace $Q$ can be written as

$$(Ay, y)_{(2.6)} = \sigma_1 z_1^2 + \sigma_2 z_2^2 + \ldots + \sigma_{n-k} z_{n-k}^2 \tag{3.6}$$

The quantity $a_0$ in Eq. (3.4) is positive, since it is equal to the sum of the squares of all the possible determinants of order $k$ which can be constructed out of the columns of the matrix $B$.

According to the Liénard-Chipart test, the roots of Eq. (3.4) are positive only if

$$a_1 > 0, \quad a_2 > 0, \ldots, \quad a_{n-k} > 0 \tag{3.7}$$

It is clear that these conditions are also sufficient (this is easy to prove indirectly). The statement of the theorem follows from (3.7) with allowance for (3.5) and (3.6).

Making use of the relations

$$a_{n-k} = D^* |_{\sigma=0} = \Delta, \quad a_{n-k-\nu} = \frac{(-1)^\nu}{\nu!} \frac{d^\nu D^*}{d\sigma^\nu}\bigg|_{\sigma=0} \quad (\nu = 1, 2, \ldots, n-k-1)$$

which follow in self-evident fashion from (3.4), we can express conditions (3.4) as

$$D^* |_{\sigma=0} > 0, \quad (-1)^\nu \frac{d^\nu D^*}{d\sigma^\nu}\bigg|_{\sigma=0} > 0 \quad (\nu = 1, 2, \ldots, n-k-1) \tag{3.8}$$

In the case $k \geqslant 1$ steady motions (2.3) and the left sides of the inequalities occurring in (2.8), (3.1), or (3.8) of these steady motions can be regarded as functions of $c_1^\circ, c_2^\circ, \ldots, c_k^\circ$ which are continuous at least for $\Delta \neq 0$ by virtue of the familiar implicit function theorem. The condition $\Delta \neq 0$ also implies the independence of Eqs. (2.6).

If conditions (2.8) or (3.1) are fulfilled at some point of the parameter space $c_1^\circ, c_2^\circ, \ldots, c_k^\circ$, then motions (2.3) are stable for all $c_1^\circ, c_2^\circ, \ldots, c_k^\circ$ from the domain defined by the condition $\Delta > 0$ which contains the point in question. In fact, with continuous variation of the parameters conditions (3.1) can be violated only if one of the roots of Eq. (3.4) vanishes. This can occur only when $a_{n-k} = \Delta = 0$. This fact is closely related to the Poincaré-Chetaev theory of bifurcation of equilibria [4].

**4.** Let us consider the steady motions of a solid body having a single fixed point and containing a cavity completely filled with a homogeneous incompressible ideal or viscous fluid, as well as the stability of these motions in a central gravitational field (*).

*) The same problem for the case of a solid body was solved by the authors of [15] and [16].

Let us introduce the movable rectangular coordinate system $Ox_1x_2x_3$ with its origin at the fixed point $O$ of the body (which lies at the distance $R$ from the attracting center $N$) and with its axes directed along the principal axes of the ellipsoid of inertia of the system for the point $O$. For simplicity of calculation we assume that the principal axes of the ellipsoid of inertia of the fluid for the point $O$ coincide with the axes $x_1$, $x_2$, $x_3$. We also introduce the following notation: $A_i$, $B_i$, $C_i$ are the moments of inertia with respect to the axes $x_i$ of the body, fluid, and the system as a whole, respectively; $\omega_i$, $G_i$, $g_i$ are the projections on the axes $x_i$ of the instantaneous velocity vector of the body and of the kinetic moment vectors with respect to the point $O$ of the fluid in its absolute and relative motions; $u_i$ are the projections on the same axes of the relative velocity vector $\mathbf{u}$ of the fluid particle with the coordinates $x_1$, $x_2$, $x_3$; $\tau$ is the volume of the cavity, $\rho$ is the density of the fluid, $\mu$ is the coefficient of viscosity, $M$ and $x_i°$ are the mass and the coordinate of the centroid of the system, $g$ is the gravitational acceleration at the distance $R$ from the attracting center, $\gamma_i$ are the direction cosines of the "vertical" $NO$ relative to the axes $x_i$, and

$$\Gamma = \gamma_1^2 + \gamma_1^2 + \gamma_3^2 = 1$$

The kinetic [6] and potential energies of the system can now be written as

$$T = \frac{1}{2}\sum_{i=1}^{3}\left(A_i\omega_i^2 + \frac{Gi^2}{B_i} + w_i^2\right), \qquad \Pi \quad \sum_{i=1}^{3}\left(Mgx_i°\gamma_i + \frac{3g}{2R}\, C_i\gamma_i^2\right)$$

$$G_1 = B_1\omega_1 + g_1, \qquad w_1^2 = \int_\tau \rho\,[u_1 + (\omega_2 - G_2 / B_2)\,x_3 - (\omega_3 - G_3 / B_3)\,x_2]^2 d\tau \qquad (123)$$

The kinetic energy and kinetic moment theorems yield the relations

$$\frac{dH}{dt} = \frac{d}{dt}(T + \Pi) = -\mu\int_\tau (\operatorname{rot}\mathbf{u})^2\,d\tau$$

$$K = \sum_{i=1}^{3}(A_i\omega_i + G_i)\,\gamma_i = \text{const}$$

For

$$n = 12,\ k = 2,\ U_0 = H,\ U_1 = K,\ U_2 = \Gamma,\ \lambda_1 = -\omega$$

$$\lambda_2 = -\sigma, \quad x_i = \omega_i, \quad x_{3+i} = G_i, \quad x_{6+i} = \gamma_i, \quad x_{9+i} = w_i \quad (i = 1, 2, 3)$$

equations of steady motions (2.2) have the solutions ($\omega$ and $\sigma$ are arbitrary constants)

$$\omega_1 = \omega\gamma_1,\ G_1 = \omega B_1\gamma_1,\ w_1 = 0 \quad (u_1 = 0) \qquad (4.1)$$

$$[\sigma + (\omega^2 - \varkappa^2)\,C_1]\,\gamma_1 = Mgx_1° \qquad (\varkappa^2 = 3g / R) \quad (123)$$

describing the steady rotations of the system as a whole about the "vertical" $\boldsymbol{\gamma}$ at the constant angular velocity $\omega$. The axes of rotation in the body lie on the Staude cone

$$x_1°\,(C_2 - C_3)\,\gamma_2\gamma_3 + x_2°\,(C_3 - C_1)\,\gamma_3\gamma_1 + x_3°\,(C_1 - C_2)\,\gamma_1\gamma_2 = 0$$

as in the case of a solid body [15].

Stability conditions (2.8) $\Delta_i > 0$ $(i = 3, \ldots, 12)$ of steady motions (4.1) reduce to the two following inequalities:

$$\Omega L > 0, \quad 4\Omega\omega^2 L + \Omega^2 SJ > 0 \qquad (4.2)$$

These inequalities are the same as those obtained for a solid body in [15].

Here

$$L = \sum_{(123)}(\lambda - C_1)(C_2 - C_3)^2\gamma_2^2\gamma_3^2, \quad S = \sum_{(123)}(\lambda - C_2)(\lambda - C_3)\,\gamma_1^2$$

$$J = C_1\gamma_1^2 + C_2\gamma_2^2 + C_3\gamma_3^2, \quad \Omega = \omega^2 - \varkappa^2 = -\sigma / \lambda$$

and the summation symbol accompanied by (123) means that the two other terms can be obtained from the term written out by permuting the subscripts 1, 2, 3.

Thus, conditions (4.2) are sufficient conditions of stability of steady rotations (4.1) of a body with a fixed point and a cavity completely filled with an ideal or viscous fluid in a central gravitational field with respect to the quantities $\omega_i$, $G_i$, $\gamma_i$, $w_i$ $(i = 1, 2, 3)$.

However, the computations can be simplified considerably as indicated in Note 4.

The altered potential energy $W^*$ is given by [5, 6, 13]

$$W^* = \Pi + \frac{k_0^2}{2J}$$

where $J$ is the moment of inertia of the system with respect to the "vertical" $NO$, and $k_0$ is the value of the constant area integral for the steady motion.

The fixed points of the function $W^*$ correspond to the steady rotations of the system as a whole about the "vertical" $\gamma$ at the constant angular velocity $\omega$. Recalling the geometric relation $\Gamma = 1$, we infer from this that expressions (4.1) are valid.

The conditions for the minimum of the function $W^*$ under the condition $\Gamma = 1$ are reducible to the requirement of positiveness of the principal diagonal third- and fourth-order minors of the determinant

$$\Delta = -\det \| e_{ij} \|_{i,\, j=1}^{i,\, j=4}, \quad e_{ij} = e_{ji}$$

where

$$e_{11} = 0, \quad e_{12} = \gamma_1, \qquad e_{13} = \gamma_2, \qquad e_{14} = \gamma_3$$

$$e_{1+i,\, 1+j} = 4\omega^2 C_i C_j \gamma_i \gamma_j J^{-1} + \delta_{ij} (\lambda - C_i)\,\Omega \quad (i, j = 1, 2, 3)$$

and where the $\delta_{ij}$ are Kronecker deltas.

Computing these minors and making use of Theorem 4, we obtain the following conditions of stability of steady rotations (4.1):

$$4\omega^2 J^{-1} \sum_{(123)} (C_2 - C_3)^2 \gamma_2^2 \gamma_3^2 + \Omega \sum_{(123)} (\lambda - C_1)(\gamma_2^2 + \gamma_3^2) > 0 \tag{4.3}$$

$$4\omega^2 \Omega J^{-1} \sum_{(123)} (\lambda - C_1)(C_2 - C_3)^2 \gamma_2^2 \gamma_3^2 + \Omega^2 \sum_{(123)} (\lambda - C_2)(\lambda - C_3)\gamma_1^2 > 0$$

which possess the advantage over conditions (4.2) because they do not degenerate for

$$\sum_{(123)} (C_2 - C_3)^2 \gamma_2^2 \gamma_3^2 = 0$$

Conditions (4.3) and (4.2) are equivalent except in this case.

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# PERIODIC SOLUTIONS OF SECOND ORDER DYNAMIC SYSTEMS CLOSE TO PIECE-WISE HAMILTONIAN SYSTEMS

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We show the conditions which must be satisfied by the approximating functions, in order that the result known for the nearly Hamiltonian systems with the analytic right sides [1] would also hold for the systems with piece-wise analytic right sides.

Theorem. Let $H(x, y) = h$ be a family of closed curves $C_h$ dependent on the parameter $h$, and matched from segments $H_i(x, y) = h$ on the intervals $x_i \leqslant x \leqslant x_{i+1}$. Functions $H_i(x, y)$ are analytic in each of their arguments.

Then a unique limit cycle exists in the neighborhood of the closed curve $C_{h_0}$, for the